数理解析研究所講究録
345巻 1979年 29-35

Non-null Distribution of the Largest Latent Root in Multivariate Analysis

統計数理研究所 杉山 高一

1. Summary. A study is made of the non-null cumulative distribution function (cdf) of the largest latent root which arises in testing of equality of two covariance matrices. This result in the null case reduces to the formula given by Sugiyama [14]. The non-central cdf of the largest latent root in the MANOVA situation is also discussed.

2. Non-null distribution of the largest latent root for test of equality of two covariance matrices.

Let $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ be independently distributed as Wishart $W(p,n_1,\underset{\sim}{\Sigma}_1)$ and $W(p,n_2,\underset{\sim}{\Sigma}_2)$, respectively, Let the latent roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ and $\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1}$ be $f_1,\ldots,f_p$ and $\omega_1,\ldots,\omega_p$, respectively, such that $\infty > f_1 > \ldots > f_p > 0$ and $\infty > \omega_1 \geqslant \ldots \geqslant \omega_p > 0$. Then the distribution of the latent roots f_i depends on the latent roots ω_i and is

(1) $g(f_1,\ldots,f_p)$

$= \text{Const}\ {}_1F_0[\tfrac{1}{2}(n_1+n_2);-\underset{\sim}{\Omega}^{-1},\underset{\sim}{F}]|\underset{\sim}{F}|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}\prod_{i<j}(f_i-f_j)$

[cf.[7],eq.65]

where $\underset{\sim}{F} = \text{diag}(f_i)$, $\underset{\sim}{\Omega} = \text{diag}(\omega_i)$,

$\text{Const}=|\underset{\sim}{\Omega}|^{-n_1/2}[\pi^{p^2/2}\Gamma_p(\tfrac{1}{2}(n_1+n_2))/\Gamma_p(\tfrac{1}{2}p)\Gamma_p(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2)]$,

and

$${}_1F_0(\tfrac{1}{2}(n_1+n_2);-\underset{\sim}{\Omega}^{-1},\underset{\sim}{F})=\int_{O(p)}{}_1F_0(\tfrac{1}{2}(n_1+n_2);-\underset{\sim}{\Omega}^{-1}\underset{\sim}{H}\underset{\sim}{F}\underset{\sim}{H}')(d\underset{\sim}{H}).$$

Since

(2) ${}_2F_1(\alpha,\beta;\gamma;\underset{\sim}{Z})$

$= |I-\underset{\sim}{Z}|^{-\beta}{}_2F_1(\gamma-\alpha,\beta;\gamma;-\underset{\sim}{Z}(\underset{\sim}{I}-\underset{\sim}{Z})^{-1})$,

then ${}_1F_0(\tfrac{1}{2}(n_1+n_2);-\underset{\sim}{\Omega}^{-1}\underset{\sim}{H}\underset{\sim}{F}\underset{\sim}{H}')$

$= {}_2F_1(\tfrac{1}{2}(n_1+n_2),a;a;-\underset{\sim}{\Omega}^{-1}\underset{\sim}{H}\underset{\sim}{F}\underset{\sim}{H}')$ is defined for $\underset{\sim}{F} > 0$ and $\underset{\sim}{\Omega} > 0$, whenever $a > \tfrac{1}{2}(p-1)$.

Formula (1) may be rewritten as

$$(3)\quad f(f_1,\ldots,f_p) = \text{Const}\sum_{k=0}^{\infty}\sum_{\kappa}[(-1)^k(\tfrac{1}{2}(n_1+n_2))_\kappa/k!C_\kappa(\underset{\sim}{I})]$$

$$\cdot\int_{O(p)}|\underset{\sim}{F}|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}C_\kappa(\underset{\sim}{H}'\underset{\sim}{\Omega}^{-1}\underset{\sim}{H}\underset{\sim}{F})\prod_{i<j}(f_i-f_j)(d\underset{\sim}{H}).$$

Using the result

$$(4)\quad \int_{x>\lambda_1>\ldots>\lambda_p>0}|\underset{\sim}{D}_\lambda|^{t-(p+1)/2}C_\kappa(\underset{\sim}{R}\underset{\sim}{D}_\lambda)\prod_{i<j}(\lambda_i-\lambda_j)\prod_{i=1}^{p}d\lambda_i$$

$$= x^{pt+k}\left(\frac{\Gamma_p(\tfrac{1}{2}p)}{\pi^{p^2/2}}\right)\frac{\Gamma_p(t,\kappa)\Gamma_p(\tfrac{1}{2}(p+1))}{\Gamma_p(t+\tfrac{1}{2}(p+1),\kappa)}C_\kappa(\underset{\sim}{R})$$

[cf.[2]eq.22 and [13]eq.2.1]

where $\underset{\sim}{D}_\lambda = \mathrm{diag}(\lambda_i)$, $\Gamma_p(a;\kappa) = \Gamma_p(a)(a)_\kappa$,
and $\underset{\sim}{R}$ is a positive definite symmetric matrix,
we obtain the cdf of the largest latent root f_1 from
(3)

$$P(f_1<x)=[\Gamma_p(\tfrac{1}{2}(n_1+n_2))\Gamma_p(\tfrac{1}{2}(p+1))/\Gamma_p(\tfrac{1}{2}n_2)\Gamma_p(\tfrac{1}{2}n_1+\tfrac{1}{2}(p+1))]$$

(5)

$$\cdot|x\Omega^{-1}|^{\frac{1}{2}n_1}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2}(n_1+n_2))_\kappa(\frac{1}{2}n_1)_\kappa C_\kappa(-x\underset{\sim}{\Omega}^{-1})}{(\frac{1}{2}n_1+\frac{1}{2}(p+1))_\kappa k!}\ .$$

Applying (2) to (5) we get

(6) $$P(f_1<x)=[\Gamma_p(\tfrac{1}{2}(n_1+n_2))\Gamma_p(\tfrac{1}{2}(p+1))/\Gamma_p(\tfrac{1}{2}n_2)\Gamma_p(\tfrac{1}{2}n_1+\tfrac{1}{2}(p+1))]$$
$$\cdot|x\Omega^{-1}(I+x\Omega^{-1})^{-1}|^{n_1/2}$$
$$\cdot{}_2F_1(\tfrac{1}{2}(-n_2+p+1),\tfrac{1}{2}n_1;\tfrac{1}{2}n_1+\tfrac{1}{2}(p+1);x\underset{\sim}{\Omega}^{-1}(I+x\underset{\sim}{\Omega}^{-1})^{-1}).$$

When $\underset{\sim}{\Omega}=\underset{\sim}{I}$, (6) reduces to the expression given by Sugiyama [14] (In [14] eq.1.7 and 1.8, $1/(1+y)$ is corrected to $y/(1+y)$). We note from the generalization of the Gauss formula ([14], eq.1.13) that $P(f_1<x)\to 1$ when $x\to\infty$.

Pillai and Sugiyama [12] derived (6) in terms of a doubly infinite sum, and Khatri [9] expressed the probability density function by a ${}_3F_2$ hypergeometric function. Under the assumption that $\frac{1}{2}(n_2-p-1)$ is an integer, Constantine and Venables [4] gave another polynomial expression.

Since

$${}_2F_1(\alpha,\beta;\gamma;\underset{\sim}{Z}) = |\underset{\sim}{I}-\underset{\sim}{Z}|^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;\underset{\sim}{Z}),$$

(6) may be rewritten as a positive series by

(7) $$P(f_1<x)=[\Gamma_p(\tfrac{1}{2}(n_1+n_2))\Gamma_p(\tfrac{1}{2}(p+1))/\Gamma_p(\tfrac{1}{2}n_2)\Gamma_p(\tfrac{1}{2}n_1+\tfrac{1}{2}(p+1))]$$
$$\cdot|x\underset{\sim}{\Omega}^{-1}|^{n_1/2}|\underset{\sim}{I}+x\underset{\sim}{\Omega}^{-1}|^{-(n_1+n_2)/2}$$
$$\cdot{}_2F_1(\tfrac{1}{2}(n_1+n_2),\tfrac{1}{2}(p+1);\tfrac{1}{2}n_1+\tfrac{1}{2}(p+1);x\underset{\sim}{\Omega}^{-1}(\underset{\sim}{I}+x\underset{\sim}{\Omega}^{-1})^{-1}).$$

Theorem 1. Let $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ be independently distributed as Wishart $W(p,n_1,\underset{\sim}{\Sigma}_1)$ and $W(p,n_2,\underset{\sim}{\Sigma}_2)$ respectively. Then the cdf of the largest latent root of $|\underset{\sim}{S}_1-f\underset{\sim}{S}_2| = 0$ is given by (6), or equivalently (7).

3. Non-central distribution of the largest root in the MANOVA case.

Let $\underset{\sim}{X}$ be a $(p\times n_1)$ matrix variate $(p\leq n_1)$ and $\underset{\sim}{Y}$ a $(p\times n_2)$ matrix variate $(p\leq n_2)$ and the columns be all independently normally distributed with covariance matrix $\underset{\sim}{\Sigma}$, $E(\underset{\sim}{X}) = \underset{\sim}{M}$ and $E(\underset{\sim}{Y}) = \underset{\sim}{O}$. Then the distribution of the latent roots f_i $(\infty>f_1>\dots>f_p>0)$ of $|\underset{\sim}{X}\underset{\sim}{X}'-f\underset{\sim}{Y}\underset{\sim}{Y}'|=0$ is

(8) $g(f_1,\dots,f_p)$

$$=\text{Const}\ |\underset{\sim}{I}+\underset{\sim}{F}|^{-\frac{1}{2}(n_1+n_2)}{}_1F_1(\tfrac{1}{2}(n_1+n_2);\tfrac{1}{2}n_1;\tfrac{1}{2}\underset{\sim}{\Omega},\underset{\sim}{F}(\underset{\sim}{I}+\underset{\sim}{F})^{-1})$$

$$\cdot\ |\underset{\sim}{F}|^{\frac{1}{2}(n_1-p-1)}\prod_{i<j}(f_i-f_j)$$

[cf.[7]eq.73 and [2]eq.41]

where ω_i are the latent roots of $|\underset{\sim}{M}\underset{\sim}{M}'-\omega\underset{\sim}{\Sigma}| = 0$,

$\underset{\sim}{\Omega} = \text{diag}(\omega_i)$, $\underset{\sim}{F} = \text{diag}(f_i)$, and

Const$= e^{-\frac{1}{2}\text{tr}\underset{\sim}{\Omega}}(\pi^{p^2/2}\Gamma_p(\tfrac{1}{2}(n_1+n_2))/\Gamma_p(\tfrac{1}{2}p)\Gamma_p(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2))$.

Using ([2],eq.1) and ([3],eq.18), Leach [11] proves by applying the Laplace transform to $\underset{\sim}{S}$ that

(9) $|\underset{\sim}{I}-\underset{\sim}{Z}|^{-a}{}_1F_1(a;b;\underset{\sim}{S},\underset{\sim}{Z}(\underset{\sim}{Z}-\underset{\sim}{I})^{-1})$

$$=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(a)_\kappa L_\kappa^{b-\frac{1}{2}(p+1)}(\underset{\sim}{S})C_\kappa(\underset{\sim}{Z})}{(b)_\kappa k!C_\kappa(\underset{\sim}{I})}$$

Now (8) may be rewritten as

$g(f_1,\ldots,f_p)$

(10)
$$=\text{Const}\sum_{k=0}^{\infty}\sum_{\kappa}[(-1)^k(\tfrac{1}{2}(n_1+n_2))_\kappa L_\kappa^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}(\tfrac{1}{2}\underset{\sim}{\Omega})/(\tfrac{1}{2}n_1)_\kappa k!C_\kappa(\underset{\sim}{I})]$$

$$\cdot|\underset{\sim}{F}|^{\frac{1}{2}(n_1-p-1)}C_\kappa(\underset{\sim}{F})\prod_{i<j}(f_i-f_j)$$

where $L_\kappa^r(\frac{1}{2}\underset{\sim}{\Omega})$ is the generalized Laguerre polynomials. Then, using (4) as before, we obtain the cdf of the largest latent root f_1

(11) $P(f_1<x)$

$$=e^{-\frac{1}{2}\text{tr}\underset{\sim}{\Omega}}[\Gamma_p(\tfrac{1}{2}(n_1+n_2))\Gamma_p(\tfrac{1}{2}(p+1))/\Gamma_p(\tfrac{1}{2}n_2)\Gamma_p(\tfrac{1}{2}n_1+\tfrac{1}{2}(p+1))]$$

$$\cdot\, x^{\frac{1}{2}pn_1}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2}(n_1+n_2))_\kappa L_\kappa^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}(\frac{1}{2}\underset{\sim}{\Omega})C_\kappa(-x\underset{\sim}{I})}{(\frac{1}{2}n_1+\frac{1}{2}(p+1))_\kappa k!C_\kappa(\underset{\sim}{I})}.$$

Since $L_\kappa^r(\underset{\sim}{0}) = (r+\frac{1}{2}(p+1))_\kappa C_\kappa(\underset{\sim}{I})$ we obtain from (2) and (11) the same cdf given by Sugiyama [14] when $\underset{\sim}{\Omega}=\underset{\sim}{0}$.

Theorem 2. Let $\underset{\sim}{S}_1$ be a matrix having non-central Wishart distribution with n_1 degrees of freedom and matrix of non-centrality parameter $\underset{\sim}{\Omega}$, and $\underset{\sim}{S}_2$ be a matrix having the Wishart distribution with n_2 degrees of freedom. Then the cdf of the largest latent root of $|\underset{\sim}{S}_1-f\underset{\sim}{S}_2|=0$ is given by (11) which converges for $|x|<1$.

Let $f_1 = \ell_1/n_2$. When n_2 tends to infinity we obtain from (11) the cdf given by Hayakawa [6] of the largest latent root ℓ_1 of non-central Wishart matrix

$$(12)\quad P(\ell_1<y) = e^{-\frac{1}{2}\mathrm{tr}\underset{\sim}{\Omega}}[\Gamma_p(\tfrac{1}{2}(p+1))/2^{\frac{1}{2}pn_1}\Gamma(\tfrac{1}{2}n_1+\tfrac{1}{2}(p+1))]$$

$$\cdot \sum_{k=0}^{\infty}\sum_{\kappa}\frac{L_{\kappa}^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}(\tfrac{1}{2}\underset{\sim}{\Omega})}{(\tfrac{1}{2}n_1+\tfrac{1}{2}(p+1))_{\kappa}}\left(-\tfrac{1}{2}y\right)^k .$$

The non-central cdf of the largest latent root in MANOVA case has been studied by Hayakawa [5], Khatri and Pillai [10], and Pillai and Sugiyama [12] who all expressed it by double infinite sums. When $\frac{1}{2}(n_2-p-1)$ is an integer, Constantine and Venables [4] derived the cdf in a polynomial expression which is simpler. We note that these results do not converge to (12) when n_2 tends to infinity. It would be useful to find a transformation such as (2) which we can apply to (10).

REFERENCES

(1) Anderson,T.W.(1965). An Introduction to Multivariate Statistical Analysis. Wiley,New York

(2) Constantine,A.G.(1963). Some non central distribution problems in multivariate analysis. Ann.Math.Statist.34 1270-1285.

(3) Constantine,A.G.(1966). The distribution of Hotelling's Generalized T . Ann.Math.Statist.37 215-225.

(4) Constantine,A.G. and Venables,W.N.(1969). On the exact distribtion of a largest characteristic root in multivariate analysis. Ann.Math.Statist.

(5) Hayakawa,T.(1967). On the distribution of the maximum latent root of a positive definite symmetric random matrix. Ann.Inst.Statit.Math.19 1-17

(6) Hayakawa,T.(1969). On the distribution of the maximum latent latent roots and traces of two positive definite random matrices. Ann.Inst.Statist.Math.21 1-21.

(7) James, A.T. (1964). Distributions of matrix variates and latent roots derived from normal samples. Ann. Math. Statist. 35 475-501.

(8) James, A.T. (1968). Calculation of zonal polynomial coefficients by use of the Laplace-Beltrami operator. Ann.Math.Statist.39 1711-1718.

(9) Khatri, C.G. (1967). Some distribution problems connected with the characteristic root of $S_1S_2^{-1}$. Ann. Math. Statist. 38 944-948.

(10) Khatri, C.G. and Pillai, K.C.S. (1968). On the non-central distributions of two test criteria in multivariate analysis of variance. Ann. Math. Statist. 39 215-226.

(11) Leach, B.G. (1969). Bessel functions of matrix argument with statistical application. Ph.D thesis in the Univ. of Adelaide. 90-98.

(12) Pillai, K.C.S. and Sugiyama, T. (1969). Non-central distributions of the largest latent roots of three matrices in multivariate analysis. Ann. Inst. Statist. Math. 21 321-327.

(13) Sugiyama, T. (1967). On the distribution of the largest latent root of the covariance matrix. Ann. Math. Statist. 38 1148-1151.

(14) Sugiyama, T. (1967). Distribution of the largest latent root and the smallest latent root of the generalized B statistics and F statistics in multivariate analysis. Ann. Math. Statist. 38, 1152-1159.